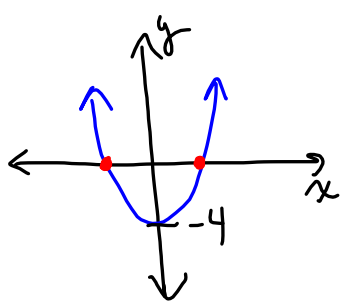
Sec. 5.6 The Fundamental Theorem of Algebra



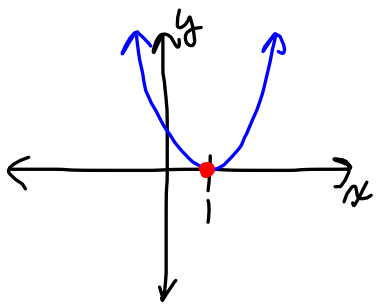
$y = x^2 - 4$

2 real roots

$x^2 - 4 = (x+2)(x-2)$

$-2, 2$

2 roots



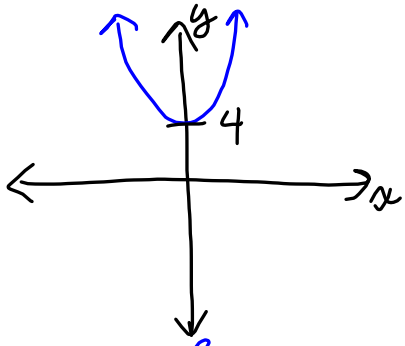
$y = x^2 - 2x + 1$

1 real root

$x^2 - 2x + 1 = (x-1)^2$

1 → multiplicity 2

2 roots



$y = x^2 + 4$

0 real roots

$x^2 + 4 = (x+2i)(x-2i)$

$-2i, 2i$

2 roots

The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x) = 0$ has exactly n roots, including multiple and complex roots.

Problem 1: What are all the complex roots of $x^5 - x^4 - 7x^3 + 7x^2 - 18x + 18 = 0$?

$\oplus$ 4 or 2 or 0

PRR: $a_0 = 18 \rightarrow \pm 1, \pm 2, \pm 3, \pm 6, \cancel{\pm 9}, \cancel{\pm 18}$

$a_n = 1$

$(-x^5 - x^4 + 7x^3 + 7x^2 + 18x + 18)$

$\ominus$ 1

	5	4	3	2	1	0
−1	1	−1	−7	7	−18	18
		−1	2	5	−12	30
	1	−2	−5	12	−30	

	x^5					
−3	1	−1	−7	7	−18	18
		−3	12	−15	24	−18
	x^4 1	−4	5	−8	6	0
1		1	−3	2	−6	
	x^3 1	−3	2	−6	0	
3		3	0	6		
	x^2 1	0	2	0		

$x = -3$

$x = 1$

$x = 3$

$x^2 + 2 = 0 \rightarrow (x + i\sqrt{2})(x - i\sqrt{2})$

$\sqrt{x^2} = \sqrt{-2}$

$x = \pm i\sqrt{2}$

$x^3 - 3x^2 + 2x - 6 = 0$

$x^2(x-3) + 2(x-3) = 0$

$(x-3)(x^2+2) = 0$

$x - 3 = 0$ $\quad x^2 + 2 = 0$

$x = 3$ $\quad -2 \quad -2$

$\sqrt{x^2} = \sqrt{-2}$

$x = \pm i\sqrt{2}$

$x = -3, 1, 3, i\sqrt{2}, -i\sqrt{2}$

Problem 2: What are the zeros of $f(x) = x^4 + 2x^3 - 4x^2 - 7x - 2$?

$+1$ $\quad = 0$

PRZ: $a_0 = -2 \rightarrow \pm 1, \pm 2$ $\qquad x^4 - 2x^3 - 4x^2 + 7x - 2$

$a_n = 1$

	x^4	x^3	x^2	x^1	x^0
	1	2	−4	−7	−2
		1	3	−1	−8
	1	3	−1	−8	

	x^4	x^3	x^2	x^1	x^0	
2	1	2	−4	−7	−2	
		2	8	8	2	
	1	4	4	1	0	$x = 2$
−1		−1	−3	−1		
	1	3	1	0		$x = -1$

$x^2 + 3x + 1 = 0$

$$x = \frac{-3 \pm \sqrt{9 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \boxed{\frac{-3 \pm \sqrt{5}}{2}}$$

$$x = -1, 2, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$$

Problem 3: $x^5 - 8x^3 - 9x + 0 = 0$

+ 1⊕ 1⊖

$-x^5 + 8x^3 + 9x$

1⊖

	5	4	3	2	1	0
0	1	0	−8	0	−9	0
		0	0	0	0	0
3	1	0	−8	0	−9	0
		3	9	3	9	
	1	3	1	3	0	

OR

$x(x^4 - 8x^2 - 9) = 0$

$x = 0$

PRR: $\pm 1, \pm 3, \pm 9$

$x = 3$

$x^3 + 3x^2 + x + 3 = 0$

$x^2(x+3) + 1(x+3) = 0$

$(x+3)(x^2+1)$

$x = -3 \qquad x^2 = -1$

$x = \pm i$

$x = 0, \pm 3, \pm i$