Sec. 5.5 Theorems About Roots of Polynomial Equations

Rational Root Theorem

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

- Integer roots are factors of a_0
- Rational roots must have reduced from $\frac{p}{q}$ where p is an integer factor of a_0 and q is an integer factor of a_n

Problem 1: What are the rational roots of $x^3 - 5x^2 - 2x + 10 = 0$?

possible RR: $a_0 = 10 \rightarrow \pm 1, \pm 2, \pm 5, \pm 10$

$a_n = 1$

$$\begin{array}{r|rrr|r} & 3 & 2 & 1 & 0 \\ 1 & 1 & -5 & -2 & 10 \\ & & 1 & -4 & -6 \\ \hline & 1 & -4 & -6 & 4 \end{array}$$

$$\begin{array}{r|rrr|r} 2 & 1 & -5 & -2 & 10 \\ & & 2 & -6 & -16 \\ \hline & 1 & -3 & -8 & -6 \end{array}$$

$$\begin{array}{r|rrr|r} 5 & 1 & -5 & -2 & 10 \\ & & 5 & 0 & -10 \\ \hline & 1 & 0 & -2 & 0 \end{array}$$

$x = 5$

$x^2 - 2 = 0$

$x^2 = 2$

$$\boxed{x = \pm\sqrt{2}, 5}$$

Problem 2:

$$8x^3+2x^2-5x+1=0$$

possible RR: $a_0 = 1 \rightarrow \pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}$

$a_n = 8 \rightarrow 1, 2, 4 8$

$$\begin{array}{r|rrrr} & 3 & 2 & 1 & 0 \\ \frac{1}{2} & 8 & 2 & -5 & 1 \\ & & 4 & 3 & -1 \\ \hline & 8 & 6 & -2 & 0 \end{array}$$

$x = \frac{1}{2}$

$$\frac{8x^2}{2}+\frac{6x}{2}-\frac{2}{2}=\frac{0}{2}$$

$$4x^2+3x-1=0$$

$4 \cdot -1 = -4$

$\boxed{-1 \cdot 4}$

$2 \cdot 2$

$$(4x-1)(x+1)=0$$

$$\boxed{x = \frac{1}{4}, -1, \frac{1}{2}}$$

Problem 3: $\frac{6x^3}{2}+\frac{2x}{2}-\frac{18}{2}=\frac{0}{2}$

PRR: $a_0 = -18 \rightarrow \pm 1, \cancel{\pm 2}, \pm 3, \cancel{\pm 6}, \pm 9, \cancel{\pm 18}$

$a_n = 6 \rightarrow 1, 2, 3, 6$ $\quad \cancel{\pm\frac{1}{2}}, \cancel{\pm\frac{3}{2}}, \cancel{\pm\frac{9}{2}}, \pm\frac{1}{3}, \cancel{\pm\frac{2}{3}}, \cancel{\pm\frac{1}{6}}$

$$3x^3+x-9=0$$

1, 3 $\qquad$ 1, 3, 9

$$\begin{array}{r|rrrr} & 3 & 2 & 1 & 0 \\ -\frac{1}{3} & 3 & 0 & 1 & -9 \\ & & -1 & +\frac{1}{3} & -\frac{4}{9} \\ \hline & 3 & -1 & \frac{4}{3} & \end{array}$$

Conjugate Root Theorem

If $P(x)$ is a polynomial with rational coefficients, then the irrational roots of $P(x) = 0$ that have the form $a + \sqrt{b}$ occur in conjugate pairs. That is, if $a + \sqrt{b}$ is an irrational root with a and b rational, then $a - \sqrt{b}$ is also a root.

Likewise, if $a + bi$ is a root, then $a - bi$ is also a root.

Problem 4: A cubic polynomial $P(x)$ has rational coefficients. If $2 + 3i$ and $\frac{2}{3}$ are two roots of $P(x) = 0$, what is one additional root?

$$2 - 3i$$

Problem 5: What is a quartic polynomial function $P(x)$ with rational coefficients so that $\underline{P(x) = 0}$ has roots 1, 3, and $1 + 2i$?

$$x = 1,\ 3,\ 1 + 2i,\ 1 - 2i$$

$$P(x) = \underbrace{(x-1)(x-3)}_{x^2 - 3x - x + 3}\ \underbrace{[x - (1+2i)]\,[x - (1-2i)]}_{(x - 1 - 2i)(x - 1 + 2i)}$$

$$(x^2 - 4x + 3)\big(x^2 \boxed{-1x} \cancel{+2ix} \boxed{-1x} + 1 - 2i \cancel{-2ix} + 2i + 4\cancel{i^2}\big)$$

$$x^2 - 4x\,3$$

$$(x^2 - 4x + 3)(x^2 - 2x + 5)$$

$$\begin{array}{r} x^4 - 2x^3 + 5x^2 \\ -4x^3 + 8x^2 - 20x \\ +3x^2 - 6x + 15 \\ \hline \end{array}$$

$$P(x) = x^4 - 6x^3 + 16x^2 - 26x + 15$$

$$[x - (1+2i)][x - (1-2i)] \qquad (a-b)(a+b) = a^2 - b^2$$

$$[\overset{a}{(x-1)} - 2i]\,[(x - 1) + 2i]$$

$$(x-1)^2 - (2i)^2$$

$$x^2 - 2x + 1 + 4\cancel{i^2}$$

$$x^2 - 2x + 5$$

Descartes' Rule of Signs

- Number of positive real roots
  = number of sign changes between consecutive coefficients of $P(x)$ OR is less than that by an even number
- Number of negative real roots
  = number of sign changes between consecutive coefficients of $P(-x)$ OR is less than that by an even number

Problem 6: What does Descartes' Rule of Signs tell you about the real roots of $P(x) = -x^4 + x^3 - 2x^2 + x + 1 = 0$?

coefficient is $-$ ✓ $+$ ✓ $-$ ✓ $+$ $+$

positive real roots 3 or 1

$$P(-x) = -(-x)^4 + (-x)^3 - 2(-x)^2 + (-x) + 1$$
$$-x^4 - x^3 - 2x^2 - x + 1$$

coefficient is $-$ $-$ $-$ $-$ $+$

negative real roots : 1

* imaginary roots: 0 or 2 imaginary

+	3	1
−	1	1
i	0	2
	4	4

Problem 7: $P(x) = 4x^5 - x^4 - x^3 + 6x^2 - 5$

$+$ ✓ $-$ $-$ ✓ $+$ ✓ $-$

positive real roots : 3 or 1

$$P(-x) = 4(-x)^5 - (-x)^4 - (-x)^3 + 6(-x)^2 - 5$$
$$= -4x^5 - x^4 + x^3 + 6x^2 - 5$$

$-$ $-$ ✓ $+$ $+$ ✓ $-$

negative real roots : 2 or 0

imaginary roots: 0 or 2 or 4

Problem 8: Write a polynomial P(x) with the following roots.

a. $-2i$ and $\sqrt{10}$

$\sqrt{x}\cdot\sqrt{x}=\sqrt{\cancel{x\cdot x}}$

x

$x=-2i,\ 2i,\ \sqrt{10},\ -\sqrt{10}$

$$P(x)=\underbrace{(x+2i)(x-2i)}\overset{a-b}{(x-\sqrt{10})}\overset{a+b}{(x+\sqrt{10})}$$

$$(x^2\cancel{-2ix}\cancel{+2ix}+4\cancel{i^2})(x^2\cancel{+\sqrt{10}x}\cancel{-\sqrt{10}x}-10)$$

$$(x^2+4)\quad(x^2-10)$$

$$x^4-10x^2+4x^2-40$$

$$P(x)=x^4-6x^2-40$$

b. $\sqrt{13}$ and $11-2i$

$x=\sqrt{13},\ -\sqrt{13},\ 11-2i,\ 11+2i$

$$P(x)=(x-\sqrt{13})(x+\sqrt{13})[x-(11-2i)][x-(11+2i)]$$

$$x^2\cancel{+\sqrt{13}x}\cancel{-\sqrt{13}x}-13\quad \overset{a+b}{((x-11)+2i)}\overset{a-b}{((x-11)-2i)}$$

$$(x^2-13)(x^2-11x\cancel{-2ix}-11x+121\cancel{+22i}$$
$$\cancel{+2ix}\cancel{-22i}+4i^2)$$

$$(x^2-13)\left(\underbrace{x^2-22x+121}_{a^2}\underbrace{-4i^2}_{-b^2}\right)$$